# PUNCTURES AND FORMALITY FOR NON-SIMPLY CONNECTED MANIFOLDS

FREDERIK BENIRSCHKE AND ALEKSANDAR MILIVOJEVIĆ

Abstract. We give a short proof that a closed orientable smooth manifold is formal if and only if it is formal after removing a point, making no assumption on the fundamental group. In the case of vanishing first Betti number, this is a corollary of a theorem of Stasheff. Furthermore, we prove that a connected sum of closed orientable smooth manifolds is formal if and only if each summand is formal, extending a classical basic result to non-simply connected manifolds. As a corollary we obtain that an orientable three–manifold is formal if its fundamental group is 1–formal. We then show that every finitely presented 1–formal group arises as the fundamental group of a formal closed four–manifold, and that virtual 1–formality implies 1–formality.

## 1. Top cell reattachment, and formality of connected sums

In [St83], Stasheff showed that the rational homotopy type of a simply connected rational Poincaré duality space, of formal dimension $n$, is determined by the rational homotopy type of its $(n-1)$–skeleton. In particular, see e.g. [LS04, Lemma 6.3], [FOT08, Theorem 3.12], a simply connected closed $n$–manifold is formal if it is formal after removing a point. Here, we give a new proof of this latter statement, while relaxing the assumption of simple connectivity to orientability; throughout, all manifolds will be assumed smooth and connected:

**Theorem 1.** *Let $M$ be a closed orientable $n$–manifold, and let $M^*$ be the manifold obtained by removing a point. If $M^*$ is formal, then so is $M$.*

*Proof.* The manifold $M^*$ is homotopy equivalent to $M$ with a small open $n$-disk removed. The double of this manifold with boundary is $M \# \overline{M}$, which has a degree one map to $M$. Taking the double of an orientable manifold with boundary preserves formality [MSZ22, Proposition 7.2 and Corollary 5.7][1]. Therefore $M \# \overline{M}$ is formal. Furthermore, formality is preserved under non-zero degree maps [MSZ23], so we conclude that $M$ is formal. □

The converse, and more, is true [LS04, Lemma 6.5]. Namely, the following holds, which we prove here for completeness, using the results of Mačinic:

**Lemma 2.** [LS04, Lemma 6.5] *If a space $X$ is formal, then each of its skeleta is formal.*

2020 *Mathematics Subject Classification.* 55P62.

*Key words and phrases.* Rational homotopy theory, Poincaré duality, formality, 1–formality.

[1]In [MSZ22, Proposition 7.2], the statement includes the assumption that the considered manifold with boundary is the thickening of a complex in Euclidean space of a given dimension, but the argument does not require that, and only uses that we have a manifold with boundary, which then deformation retracts onto its $(n-1)$–skeleton. In our case, we have $M^*$ deformation retracting onto the $(n-1)$–skeleton of $M$.

*Proof.* The inclusion of the $k$–skeleton $X^{(k)} \hookrightarrow X$ induces an isomorphism on homology in degrees $\leq k-1$ and a surjection in degree $k$. Since $X$ is formal, it follows from [M10, Corollary 2.4] that $X^{(k)}$ is $(k-1)$–formal in the sense of loc. cit. Then, by [M10, Proposition 3.4], $X^{(k)}$ is formal. □

Since a closed smooth $n$–manifold admits a cell decomposition with a single $n$–cell, the manifold obtained by removing a point is homotopy equivalent to the $(n-1)$–skeleton.

**Corollary 3.** *A closed orientable manifold is formal if and only if it is formal after removing a point.*

The following is also useful to record for the non-simply connected case:

**Corollary 4.** *The connected sum of two closed orientable manifolds is formal if and only if each summand is formal.*

*Proof.* Let $M, N$ be closed orientable $n$–manifolds. If the connected sum $M\#N$ is formal, then using [MSZ23] we see that $M$ and $N$ are formal, since there are degree one maps from $M\#N$ to $M$ and $N$ given by collapsing one of the summands. Conversely, if $M$ and $N$ are formal, then so are their $(n-1)$–skeleta $M^{(n-1)}$ and $N^{(n-1)}$. Then their wedge sum $M^{(n-1)} \vee N^{(n-1)}$ is formal (see e.g. the model for the wedge sum given in [FOT08, Example 2.47]), and since $M^{(n-1)} \vee N^{(n-1)} \simeq (M\#N)^{(n-1)}$, we conclude by Theorem 1 that $M\#N$ is formal. □

## 2. Three–manifolds, four–manifolds, and 1–formality

Instead of formality, one can consider notions of partial formality of a space, tantamount to formality of appropriate corresponding skeleta [M10]. As a special case of particular interest, one says a space $X$ is 1–formal if there is a zigzag of morphisms of commutative differential graded algebras between the piecewise-linear forms $(A_{PL}(X), d)$ and the cohomology $(H(X), 0)$, each of which induces an isomorphism on first cohomology and an injection on second cohomology. The 1–formality of a space only depends on its fundamental group, and so we can say a group $G$ is 1–formal if some (equivalently, every) space $X$ with $\pi_1(X) \cong G$ is 1–formal. For a survey of this notion we refer the reader to [PS09].

Strictly stronger notions of partial formality for spaces, in particular of 1–formality, are considered in [FM05], and a comparison with the notions above is discussed in [M10]. In the stronger sense of [FM05], a 1–formal closed orientable manifold of dimension $\leq 4$ is formal [FM05, Theorem 3.1]. A group with abelianization of rank $b_1 \leq 1$ is 1–formal (and there are non-1–formal groups with $b_1 \geq 2$), and it turns out that a closed orientable manifold of dimension $\leq 4$ with first Betti number $b_1 \leq 1$ is 1–formal in the stronger sense of [FM05] and hence formal [FM05a, Proposition 4.1]. Therefore, the low values of $b_1$ ensuring 1–formality of a group also ensure formality of the ambient closed orientable $n$–manifold provided $n \leq 4$. We will now see that, in the case of closed orientable manifolds of dimension three, the two notions of 1–formality coincide regardless of $b_1$:

**Theorem 5.** *If the fundamental group of a closed orientable three–manifold $M$ is 1-formal, then $M$ is formal.*

*Proof.* Removing a point from $M$ produces a manifold with the homotopy type of the two–skeleton $M^{(2)}$ of $M$. Since the inclusion $M^{(2)} \hookrightarrow M$ induces an isomorphism on fundamental groups, $M^{(2)}$ is 1–formal and hence formal by [M10, Proposition 3.4]. Then, by Theorem 1, we conclude that $M$ is formal. □

This result is immediate for non-closed three–manifolds, as their homotopy types are those of their two–skeleta.

**Remark 6.**
(1) One would expect the above from a consideration of Massey products as well. Indeed, all Massey products landing in degree two on a 1–formal space vanish, see e.g. [S23, Proposition 3.15]. For Massey products landing in degree three, one can apply the argument given in [CFM08, comments after Lemma 7] to show that on any rational Poincaré duality space, Massey products landing in top degree vanish.
(2) We also point out the fact that the fundamental group of a closed orientable three–manifold determines its homotopy type; see e.g. [AFW15, Theorem 2.3]. One could then likewise prove Theorem 5 by exhibiting a formal closed orientable three–manifold with a prescribed 1–formal fundamental group.
(3) For closed orientable manifolds of dimension $\geq 5$, it is not true that 1–formality implies formality. Indeed, one can for example consider the five–dimensional Heisenberg manifold, obtained by taking the total space of the principal circle bundle over a four–torus whose Euler class is represented by the standard symplectic form on the four–torus. This manifold is 1–formal but not formal, see [M10, Remark 5.4]. One obtains such examples in higher dimensions by taking the product of this manifold with a sphere of the appropriate dimension (in the six–dimensional case, note that the product of a 1-formal group and the integers is a 1-formal group). We point the reader also to [FM05a], [FM05b, Theorem 1.1], where the authors construct non-formal closed orientable manifolds of dimensions five and six, both of which have fundamental group $\mathbb{Z}$ and are hence 1–formal. It is unclear whether one should expect 1–formal closed orientable four–manifolds to be formal. Generally, 1–formality is equivalent to formality of the 2–skeleton, and as discussed in [KT91, p. 362], having a formal half-dimensional skeleton does not ensure formality of a closed orientable manifold.

Consider now the following question [FM05a, p.129]:

**Question 7.** Given a finitely presented group $\Gamma$ and an integer $n$ with

$$n \geq \max(3, 7 - 2b_1(\Gamma))^2,$$

are there always non-formal $n$-manifolds $M$ with fundamental group $\Gamma$?

The assumption throughout loc. cit. is that the manifolds under consideration are closed, orientable, and connected. For the case of $n = 3$, the answer is generally negative, as not every finitely presented group is the fundamental group of a closed three–manifold (e.g. $\mathbb{Z}^k$ for $k \neq 0, 1, 3$). But even within the class of $\Gamma$ realized as the fundamental group of a closed orientable three–manifold, not every such $\Gamma$ is the fundamental group of a closed orientable non-formal three–manifold. Indeed, this follows from our Theorem 5, or simply Remark 6(2) applied to any finitely generated 1–formal $\Gamma$. For example, we can

[2] In loc. cit., $2b_1(\Gamma) - 7$ is written instead of $7 - 2b_1(\Gamma)$, but $7 - 2b_1(\Gamma)$ is in line with Theorem 1 therein. In either case we provide counterexamples. Note that for the $2b_1(\Gamma) - 7$ variant, $n = 3, 4$ and $\Gamma = \{0\}, \mathbb{Z}$ would provide negative answers as well.

take $\Gamma$ to be the free product of $k \geq 2$ copies of $\mathbb{Z}$. The connected sum of $k$ copies of $S^1 \times S^2$ has this as its fundamental group. We have $b_1(\Gamma) \geq 2$, and so $n = 3$ satisfies $n \geq \max(3, 7 - 2b_1(\Gamma))$, giving a negative answer to Question 7.

Though we are not able to determine whether a 1–formal closed orientable four–manifold is necessarily formal, we can show that given any finitely presented group $\Gamma$, there is a closed orientable formal four–manifold with $\Gamma$ as its fundamental group. For context, let us first note the following easier result:

**Proposition 8.** *Let $\Gamma$ be a finitely presented 1–formal group. Then there is a formal closed connected orientable manifold $M$ such that $\pi_1(M) \cong \Gamma$.*

*Proof.* Take a finite two–dimensional cell complex $K$ embedded in $\mathbb{R}^n$ for some $n$, whose fundamental group is $\Gamma$. This is a formal space by [M10, Proposition 3.4]. Now take a thickening of the two–complex in $\mathbb{R}^n$ in the sense of Wall [W66]; its existence is guaranteed as long as $n \geq 5$. This thickening is an orientable manifold $N$ with boundary $\partial N$ such that the inclusion of the boundary $\partial N \hookrightarrow N$ is an isomorphism on fundamental groups. Since $N$ is homotopy equivalent to $K$, it is also formal. Therefore its double is a formal closed connected orientable manifold, which is formal as used in Theorem 1. Since $\partial N \hookrightarrow N$ induces an isomorphism on fundamental groups, by Seifert–van Kampen, the fundamental group of the double is isomorphic to $\Gamma$. □

The manifolds obtained by the above procedure have dimension at least five. We can improve on this as follows:

**Proposition 9.** *Let $\Gamma$ be a finitely presented 1–formal group. Then there is a formal closed connected orientable four–manifold $M$ such that $\pi_1(M) \cong \Gamma$.*

*Proof.* Take a finite two–dimensional cell complex with fundamental group $\Gamma$. By a theorem of Stallings [S65], see [DR93, Corollary], there is a homotopy equivalent polyhedron that embeds in $\mathbb{R}^4$. From the proof of Stallings' theorem given in [DR93] (see also [S65, Corollary 4.2]), we may take the $P$ to be a subpolyhedron of $\mathbb{R}^4$, which is furthermore finite. Now, $P$ might not be two–dimensional, but we can consider only the faces of dimension $\leq 2$ and thus obtain a two–dimensional polyhedron with fundamental group $\Gamma$, which embeds in $\mathbb{R}^4$. Being two–dimensional, this smaller polyhedron is formal as $\Gamma$ is 1-formal [M10, Proposition 3.4]. Let us thus work with this two–dimensional polyhedron instead, which we will refer to simply as $P$.

Now, we take a PL regular neighborhood $N$ of $P$ in $\mathbb{R}^4$ [RS, Chapter 3], see also [Cu61, Section 3]. The inclusion of the boundary $\partial N \hookrightarrow N$ induces a surjection on fundamental groups. Indeed, since $P$ has codimension two in $N$, by general position we may move any loop in $P$ off of $P$ inside $N$. Furthermore, since $N$ has the structure of a mapping cylinder (see e.g. [M76, p.417]), we can move the loop to $\partial N$.

By Hirsch–Mazur smoothing theory, we may smooth $(N, \partial N)$ to a smooth compact four–manifold $M$ with boundary [M11, Theorem 2]. Since $M$ has the homotopy type of $N$, which in turn has the homotopy type of $P$ (being a regular neighborhood), it is formal. As used in the proof of Theorem 1, the double of $M$ is formal as well. Furthermore, since the inclusion $\partial M \hookrightarrow M$ induces a surjection on fundamental groups, by Seifert–van Kampen we see that the double of $M$ has fundamental group $\Gamma$. □

We compare the above to the fact that every group which is the fundamental group of some complex projective variety is also the fundamental group of a complex projective surface (obtained by iterated application of the Lefschetz hyperplane theorem). It is an open problem whether the analogous statement holds for fundamental groups of compact Kähler manifolds, namely whether a Kähler group is the fundamental group of some compact Kähler surface.

We finish by recording the following property of 1–formality:

**Proposition 10.** *A finitely presented group is 1-formal if it is virtually 1–formal.*

By *virtual* 1–formality we mean the existence of a finite-index subgroup which is 1–formal. Under the additional assumption that the 1–formal subgroup is normal, this result is implicit in the work of Papadima [P82].

*Proof.* Let $\Gamma$ be a finitely presented group, with a 1–formal finite index subgroup $H$, and take a two–complex $K$ with fundamental group $\Gamma$. Now, thicken $K$ to an orientable manifold $N$ with boundary in some $\mathbb{R}^n$, and consider the finite cover $N' \to N$ corresponding to the subgroup $H$. This map sends $\partial N'$ to $\partial N$, and hence extends to a finite cover of the double of $N'$ to the double of $N$. Since $N'$ has the homotopy type of a two–complex, it is formal since $H$ is formal, so its double is formal, and hence the double of $N$ is formal by [MSZ23, Theorem A]. Since the double of $N$ retracts onto $N$, we conclude that $N$ is formal [FOT08, Example 2.88]. Alternatively, we could have taken $N$ and $N'$ to be open orientable manifolds; then the covering map is a proper map of non-zero degree in Borel–Moore homology, whence formality of $N$ follows also by [MSZ23, Theorem A]. ☐

A 1–formal group can contain finite index subgroups that are not 1–formal. For example, we can take a non-Haken closed orientable three–manifold modelled on Nil as in [BK24]. Being non-Haken, such a manifold is a rational homology three–sphere and hence formal, while being modelled on Nil provides us with a finite covering from a Heisenberg nilmanifold, which is non-formal. We now apply Theorem 5.

**Acknowledgements.** The first-named author would like to thank the University of Waterloo, and the second-named author would like to thank the University of Chicago, for their hospitality during visits where parts of this work were carried out. We would like to thank Igor Belegradek for providing the example in the last paragraph, together with Grigori Avramidi for helpful references.

## References

[AFW15] Aschenbrenner, M., Friedl, S., Wilton, H. and Friedl, S., 2015. *3–manifold groups* (Vol. 20). Zürich: European Mathematical Society.

[BK24] Bamler, R.H. and Kleiner, B., 2024. *Diffeomorphism groups of prime 3-manifolds.* Journal für die reine und angewandte Mathematik (Crelles Journal), 2024(806), pp.23-35.

[CFM08] Cavalcanti, G.R., Fernández, M. and Muñoz, V., 2008, June. *On Non-Formality of a Simply-Connected Symplectic 8-Manifold.* In AIP Conference Proceedings (Vol. 1023, No. 1, pp. 82-91). American Institute of Physics.

[Cu61] M. L. Curtis, 1961. *On 2-complexes in 4-space*, in: Topology of 3-manifolds and related topics. Proceedings of the University of Georgia Institute 1961. Englewood Cliffs, N.J.: Prentice-Hall, Inc.. 204–207

[DR93] Dranišnikov, A.N. and Repovš, D., 1993. *Embedding up to homotopy type in Euclidean space.* Bulletin of the Australian Mathematical Society, 47(1), pp.145-148.

[FOT08] Félix, Y., Oprea, J. and Tanré, D., 2008. *Algebraic models in geometry.* Oxford University Press.

[FT86] Félix, Y. and Tanré, D., 1986. *Formalite d'une application et suite spectrale d'Eilenberg-Moore.* In Algebraic Topology Rational Homotopy: Proceedings of a Conference held in Louvain-la-Neuve, Belgium, May 2–6, 1986 (pp. 99-123). Berlin, Heidelberg: Springer Berlin Heidelberg.

[FM05] Fernández, M. and Muñoz, V., 2005. *Formality of Donaldson submanifolds.* Mathematische Zeitschrift, 250, pp.149-175.

[FM05a] Fernández, M. and Muñoz, V., *The geography of non-formal manifolds.* In: Kowalski, Oldřich (ed.) et al., Complex, contact and symmetric manifolds. In honor of L. Vanhecke. Selected lectures from the international conference "Curvature in Geometry" held in Lecce, Italy, June 11–14, 2003. Boston, MA: Birkhäuser (ISBN 0-8176-3850-4/hbk). Progress in Mathematics 234, 121-129 (2005)

[FM05b] Fernández, M. and Muñoz, V., *Non-formal compact manifolds with small Betti numbers*, in Bokan, Neda (ed.) et al., Proceedings of the conference on contemporary geometry and related topics, Belgrade, Serbia and Montenegro, June 26–July 2, 2005. Belgrade: University of Belgrade, Faculty of Mathematics

[KT91] Kreck, M. and Triantafillou, G., 1991. *On the classification of manifolds up to finite ambiguity.* Canadian Journal of Mathematics, 43(2), pp.356-370.

[LS04] Lambrechts, P. and Stanley, D., 2004. *The rational homotopy type of configuration spaces of two points.* In Annales de l'institut Fourier (Vol. 54, No. 4, pp. 1029-1052).

[M10] Mačinic, A.D., 2010. *Cohomology rings and formality properties of nilpotent groups.* Journal of Pure and Applied Algebra, 214(10), pp.1818-1826.

[MSZ22] Milivojević, A., Stelzig, J. and Zoller, L., 2022. *Poincaré dualization and Massey products.* arXiv preprint arXiv:2203.15098.

[MSZ23] Milivojević, A., Stelzig, J. and Zoller, L., 2023. *Formality is preserved under domination.* arXiv preprint arXiv:2306.12364.

[M76] Miller, R.T., 1976. *Mapping cylinder neighborhoods of some ANR's.* Annals of Mathematics, 103(3), pp.417-427.

[M11] Milnor, J., 2011. *Differential topology forty-six years later.* Notices of the AMS, 58(6).

[P82] Papadima, S., 1982. *On the formality of maps.* An. Univ. Timisoara Ser. Stiint. Mat, 20.

[PS09] Papadima, S. and Suciu, A., 2009. *Geometric and algebraic aspects of 1-formality.* Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, pp.355-375.

[RS] Rourke, C.P. and Sanderson, B.J., 2012. *Introduction to piecewise-linear topology.* Springer Science & Business Media.

[S65] Stallings, J. R., 1965. *The embedding of homotopy types into manifolds.* Mimeographed notes, Princeton University Press.

[St83] Stasheff, J., 1983. *Rational Poincaré duality spaces.* Illinois Journal of Mathematics, 27(1), pp.104-109.

[S23] Suciu, A.I., 2023. *Formality and finiteness in rational homotopy theory.* EMS Surveys in Mathematical Sciences, 10(2), pp.321-403.

[W66] Wall, C.T.C., 1966. *Classification problems in differential topology—IV: Thickenings.* Topology, 5(1), pp.73-94.

University of Chicago
*Email address*: benirschke@uchicago.edu

University of Waterloo
*Email address*: amilivoj@uwaterloo.ca